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Seismic inversion based on L1-norm misfit function and total variation regularization

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A B S T R A C T

Objective: To solve the inverse problems when outliers exist in the seismic data and discontinuities such as layer boundaries need to be clearly delineated and merge the low frequency information to the inverted parameters.
Methods: L1-norm misfit function, total variation regularization, a priori information constraints, method of Lagrange multipliers, and iteratively re-weighted least squares.
Results and conclusions: Integrating the L1-norm misfit function, total variation regularization and a priori information constraints via the method of Lagrange multipliers, we create the objective function of seismic inversion to solve the inverse problems that outliers exist in the seismic data and discontinuities such as layer boundaries need to be clearly delineated. In addition, the priori information constraints ensure the inverted parameters have low frequency components.
Practice: The proposed inversion method is successfully tested on noisy synthetic seismic data with outliers and real seismic data.
Implications: If there are a small number of outliers in the seismic data, we need to do the seismic inversion in a way that minimizes their effect on the estimated parameters. However, the L2-norm misfit function is highly susceptible to even small numbers of inconsistent seismic observations. As an alternative to L2-norm, one can consider the solution that minimizes the L1-norm misfit function (L1MF) which will be more outlier-resistant, or robust, than the L2-norm solution. Of course, there are some alternative techniques to find the favorable regularization parameters. A set of good regularization parameters is the key of the seismic inversion process.

1. Introduction

Robustness is an important property in seismic inversion strategy. The potential disadvantages of computing L2-norm misfit function to solve seismic inverse problems have long been assessed. Many scholars discussed a variety of robust procedures. Claerbout and Muir (1973) give many examples to illustrate the advantages of L1-norm misfit function inversion. They point out that these advantages are based on the fact that solutions to L2 norm misfit function tend to overstate the influence of outliers which may arise from the procedural measurement error, or other reasons, in the seismic data. However, there are some disadvantages in the L1-norm optimization. The biggest one is that the inverse problem is nonlinear. In order to solve this problem, the most useful method is simplex-based method for linear programming extensions of Gaussian method (Barrowdale and Roberts, 1974; Bloomfield and Steiger, 1984). The other techniques for L1-norm minimization methods are based on the interior-point and iteratively re-weighted least square (Aster et al., 2005, 2013; Coleman and Li, 1992; Portnoy and Koenker, 1997; Scales et al., 1988; Watson, 2000). The iteratively re-weighted least square (IRLS) method is the simplest way to implement in L1-norm optimization. It is most attributed by Schlossmacher (1973), Beaton and Tukey (1974). Byrd and Pyne (1979) provided the convergence results through numerous references to the use of IRLS, and so did Bissantz et al. (2009). Huber (1996) reviewed the history of methods for finding the robust solutions and discussed a variety of robust procedures.

In addition, total variation regularization (TVR) is appropriate for the inverse problems when we expect to estimate the discontinuities which are desirable in geologic environments with abrupt changes in P-wave impedance, such as carbonate caves, salt bodies, or strong faults. Methods for TVR are discussed in the following references (Osher and Fedkiw, 2002; Rodriguez, 2013; Varela, Verdin and Sen, 2006; Vogel, 2002). Due to the band-limited nature of seismic data, we add a priori information constraints as regularization terms to ensure the inverted parameters contain low frequency information.

Integrating the L1-norm misfit function, total variation regularization and a priori information constraints via the method of Lagrange multipliers (Hansen, 1992), the objective function of seismic inversion is created. In order to find the solution of this inverse problem, we use

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IRLS strategy. The proposed method is tested on noisy synthetic seismic data with some outliers. At last, we perform this method using real seismic data to further verify its feasibility and stability.

2. Objective function

L2-norm misfit function solutions are highly sensitive to even small number of outliers which are data points highly discordant with the other seismic data. If there are some outliers in the seismic data due to incorrect measurements or other reasons, it is necessary to do the seismic inversion in a way that can minimize their effects on the inverted parameters. As an alternative to L2-norm misfit function, we consider to solve the inverse problem via minimizing the L1-norm misfit function, which is as the following

$$\|\mathbf{e}\|_1 = \|\mathbf{G}(\mathbf{m}) - \mathbf{d}\|_1 \tag{1}$$

where **e** is the vector of the residual, **d** is the vector of the seismic data, **G** is the forward operator, and **m** is the vector of the earth model parameters. For seismic inversion, **m** is the vector of P-wave impedance. The L1-norm misfit function solution will be more robust than the L2-norm one, because Eq. (1) does not square each of the terms in the misfit measurement. In Claerbout and Muir's words, the median value is more robust than the mean one (Claerbout and Muir, 1973).

The L1-norm misfit function solution is the maximum likelihood estimator for noisy seismic data corresponding to an exponential distribution

$$f(x) = \frac{1}{\sqrt{2}\sigma} e^{-\sqrt{2}|x-\mu| / \sigma} \tag{2}$$

where, x is a random variable, μ is the expectation, and σ is the standard deviation.

Seismic data distributed as exponential distributions are unusual. However, it is worthwhile to solve an inverse solution based on L1-norm misfit function rather than L2-norm misfit function. Even if most of the seismic data noise is distributed as normal distribution, there are reasons to doubt the existence of outliers.

In addition, total variation regularization is appropriate for the inverse problem when we expect to estimate the layer boundaries or edges. In the one-dimensional case, the first order TVR function is

$$TVR_1(\mathbf{m}) = \|\mathbf{T}_1\mathbf{m}\|_1 \tag{3}$$

and the second order TVR function

$$TVR_2(\mathbf{m}) = \|\mathbf{T}_2\mathbf{m}\|_1. \tag{4}$$

In Eqs. (3) and (4),

$$\mathbf{T}_1 = \begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \\ & & & & -1 & 1 \end{bmatrix} \tag{5}$$

$$\mathbf{T}_2 = \begin{bmatrix} 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 & 1 \end{bmatrix} \tag{6}$$

we can see that $\mathbf{T}_1$ is the first order finite difference operation, and $\mathbf{T}_2$ is the second order finite difference operation. For higher dimensions, $\mathbf{T}_1$ and $\mathbf{T}_2$ are often implemented as finite difference approximations to the gradient and Laplacian operators, respectively.

Here we hope to consider all the solutions with $\|\mathbf{T}_i\mathbf{m}\| < \delta$ and select the one which can minimize Eq. (1). So the inverse problem has the following form

$$\begin{cases} \min\|\mathbf{G}(\mathbf{m})\text{-}\mathbf{d}\|_1 \\ \|\mathbf{T}_i\mathbf{m}\|_1 < \delta \end{cases} \tag{7}$$

where, i = 1, 2.

Using the Lagrange multiplier technique, Eq. (7) can be turned into an unconstrained optimization problem

$$\min\|\mathbf{G}(\mathbf{m}) - \mathbf{d}\|_1 + \alpha\|\mathbf{T}_i\mathbf{m}\|_1 \tag{8}$$

where α is the regularization parameter for TVR.

Due to the band-limited nature of seismic data, a priori information constraint can be added as a regularization term to ensure the inverted parameters containing low frequency components. The contents of the a priori model **m** come from well log data, geological horizons, or other sources. In seismic inversion processing, according to the definition of reflection coefficient

$$r(t) = \frac{m(t+1) - m(t)}{m(t+1) + m(t)} \tag{9}$$

when $r(t)$ is small it can be approximated as

$$r(t) \approx \frac{\Delta m(t)}{2m(t)} \approx \frac{\partial}{\partial t} \frac{\ln m(t)}{2} \tag{10}$$

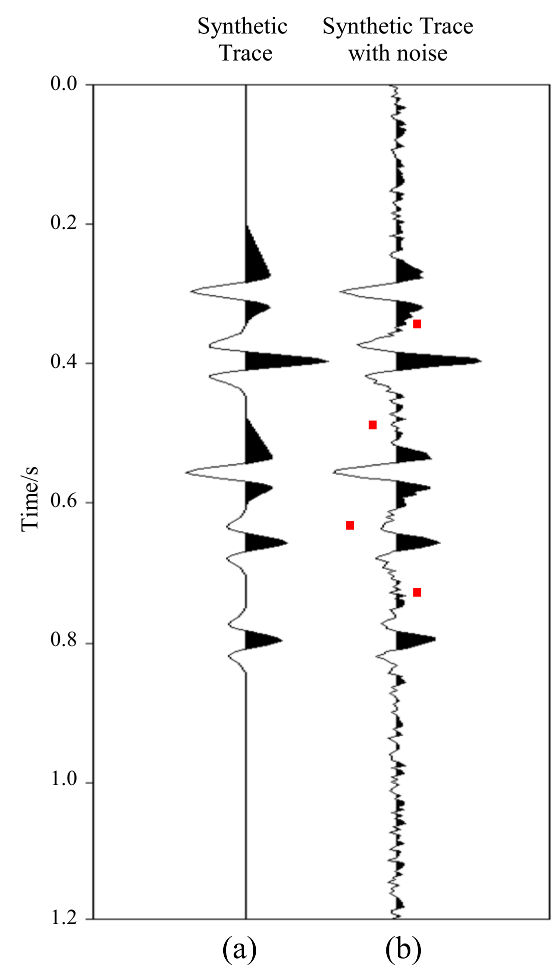

Fig. 1. (a) noise-free seismic data and (b) seismic data with 10% Gaussian random noise and outliers.

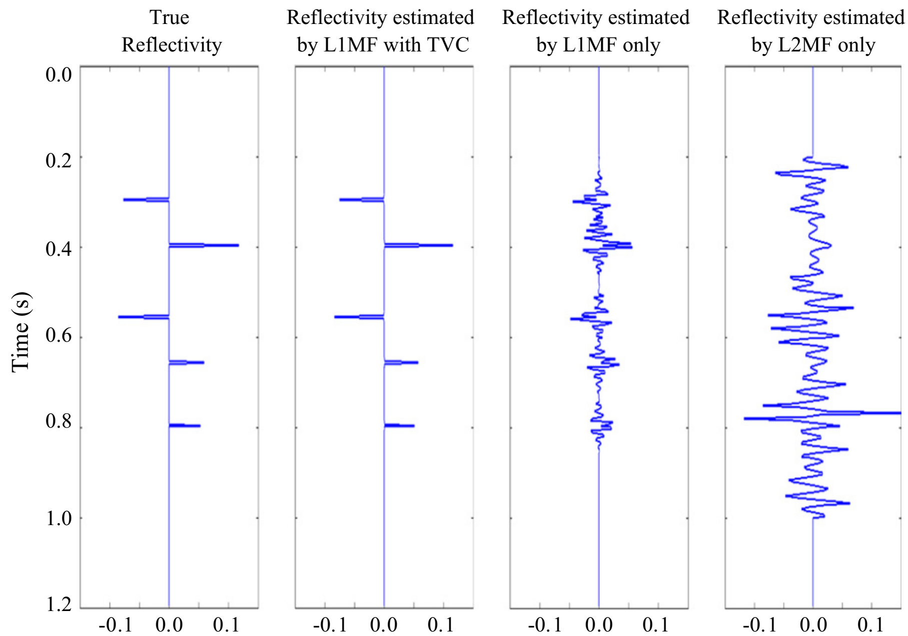

Fig. 2. Comparison between three kinds of estimated reflection coefficients and true reflectivity.

where $m(t)$ is the P-wave impedance at time t, and $r(t)$ is the P-wave reflection coefficient at time t. Taking the integral of the both sides of Eq. (10)

$$\xi_t = \frac{1}{2}\ln\frac{m(t)}{m(t_0)} = \int_{t_0}^{t} r(\eta)d\eta \tag{11}$$

where ξ_t is called the relative impedance. In the discrete version, Eq. (11) can be written as a matrix form

$$\boldsymbol{\xi} = \mathbf{Cr} \tag{12}$$

where $\boldsymbol{\xi}$ is the vector of relative impedance, $\mathbf{r}$ is the reflection coefficient sequence, and $\mathbf{C}$ is the integral operator

$$\mathbf{C} = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ \vdots & \vdots & \ddots & \\ 1 & 1 & 1 & 1 \end{bmatrix}. \tag{13}$$

Adding the a priori regularization term into Eq. (8), the objective function will be

$$f(\mathbf{m}) = \|\mathbf{G}(\mathbf{m}) - \mathbf{d}\|_1 + \alpha\|\mathbf{T}_i\mathbf{m}\|_1 + \beta\|\mathbf{Cr} - \boldsymbol{\xi}\|_1 \rightarrow \min \tag{14}$$

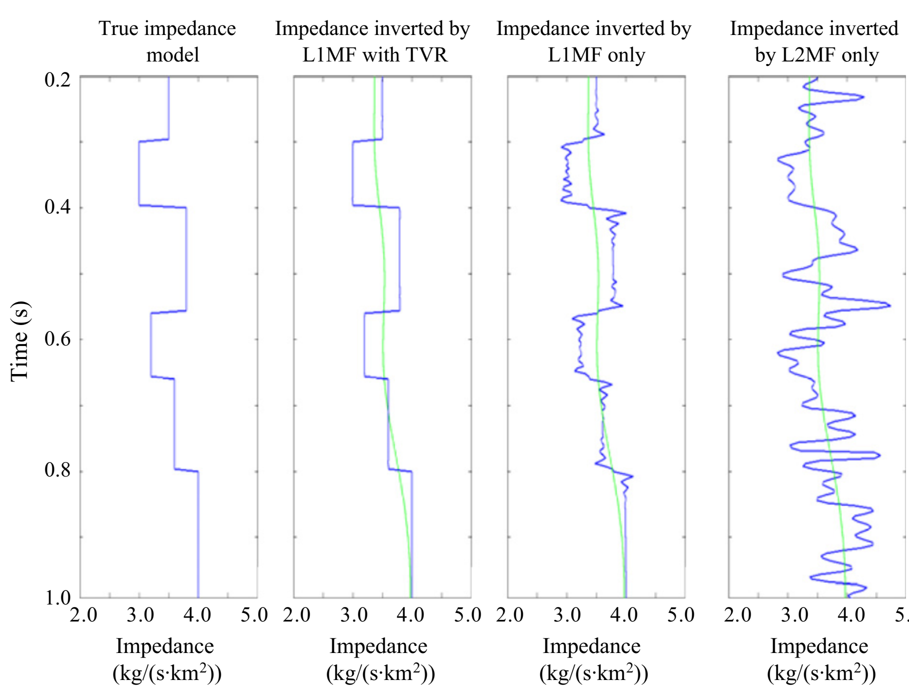

Fig. 3. Comparison between three kinds of estimated impedance and true model.

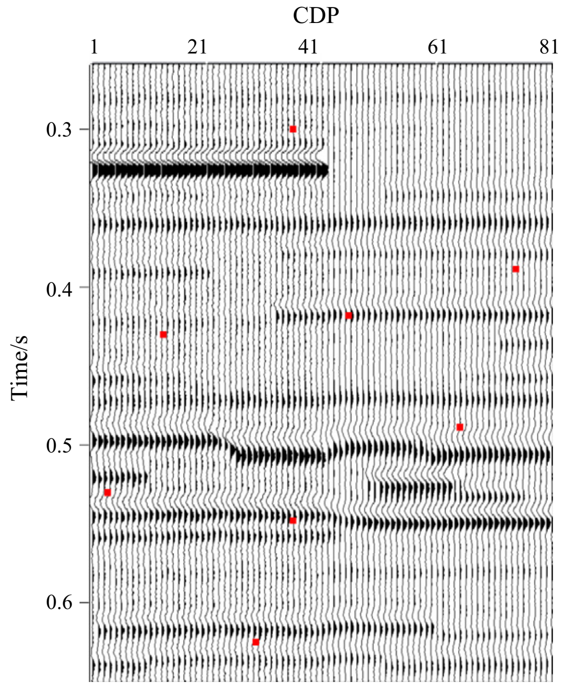

Fig. 4. Seismic data with Gaussian random noise and outliers for 2-dimensional model.

where β denotes the regularization parameter for the a priori regularization term.

In terms of Robinson convolution model, seismic signal is the convolution of reflection coefficients and a wavelet, that is

$$\mathbf{d}{=}\mathbf{G}\mathbf{r} \tag{15}$$

where, $\mathbf{G}$ represents the wavelet convolution matrix. From the definition of reflection coefficient we can see that the first order TVR for $\mathbf{m}$ is equivalent to the sparse regularization (Bruckstein et al., 2009; Candes and Tao, 2006; Candes et al., 2006a, 2006b; Claerbout and Muir, 1973) for $\mathbf{r}$, and the second order TVR for $\mathbf{m}$ is equivalent to the first order TVR for $\mathbf{r}$. So the final objective function for the inverse problem is

$$f(\mathbf{r}) = \|\mathbf{Gr}-\mathbf{d}\|_1 + \alpha\|\mathbf{r}\|_1 + \alpha\|\mathbf{T}_1\mathbf{r}\|_1 + \beta\|\mathbf{Cr}-\boldsymbol{\xi}\|_1 \rightarrow \min. \tag{16}$$

3. IRLS to solve objective function

Let

$$\mathbf{e}{=}\mathbf{Gr}-\mathbf{d} \tag{17}$$

$$\mathbf{a}{=}\mathbf{T}_1\mathbf{r} \tag{18}$$

$$\mathbf{g}{=}\mathbf{Cr}-\boldsymbol{\xi} \tag{19}$$

at point where any element of $\mathbf{e}$, $\mathbf{r}$, $\mathbf{a}$ and $\mathbf{g}$ is zero, $f(\mathbf{r})$ is not differentiable. Ignoring these non-differentiable points at first, the derivative of $f(\mathbf{r})$ is

$$\frac{\partial f}{\partial r_j} = \sum_{i=1}^{n} \frac{\partial(|e_i| + \alpha|r_i| + \alpha|a_i| + \beta|g_i|)}{\partial r_j} = \sum_{i=1}^{n}[G_{i,j}\,\mathrm{sgn}(e_i) + \alpha\,\mathrm{sgn}(r_i) + \alpha T_{1i,j}\,\mathrm{sgn}(a_i) + \beta C_{i,j}\,\mathrm{sgn}(g_i)] \tag{20}$$

where, $\mathrm{sgn}(x)$ is the signum function

$$\mathrm{sgn}(x) = \frac{x}{|x|}. \tag{21}$$

So Eq. (20) can be written as the following form

$$\frac{\partial f}{\partial r_j} = \sum_{i=1}^{n}[G_{i,j}\frac{e_i}{|e_i|} + \alpha\frac{r_i}{|r_i|} + \alpha T_{1i,j}\frac{a_i}{|a_i|} + \beta C_{i,j}\frac{g_i}{|g_i|}]. \tag{22}$$

Hence the gradient of $f(\mathbf{r})$ is

$$\nabla f = \mathbf{G}^T\mathbf{Ee} + \alpha\mathbf{Rr} + \alpha\mathbf{T}_1^T\mathbf{Aa} + \beta\mathbf{C}^T\mathbf{Bg} \tag{23}$$

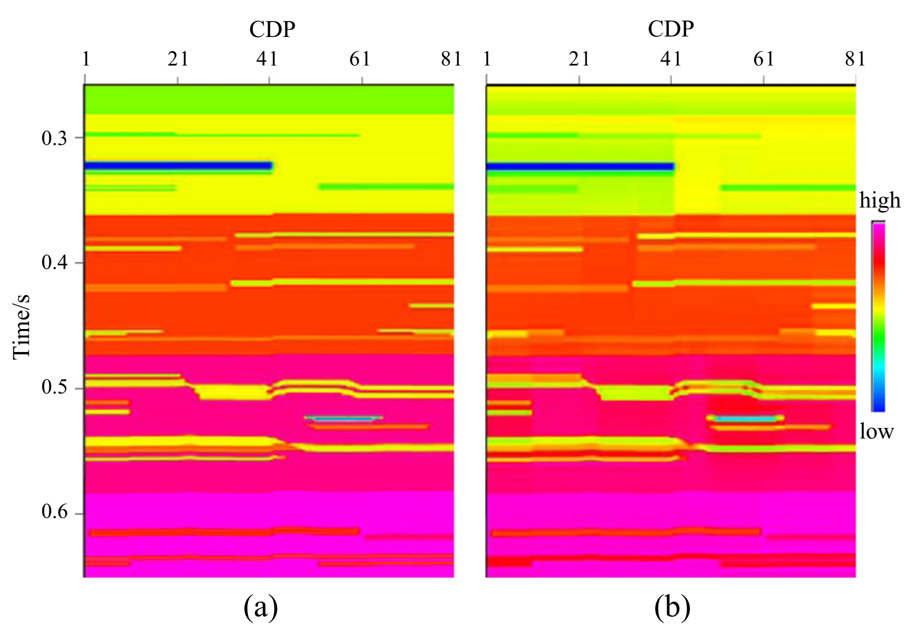

Fig. 5. Comparison between (a) the 2-dimensional true impedance model and (b) the inversion result.

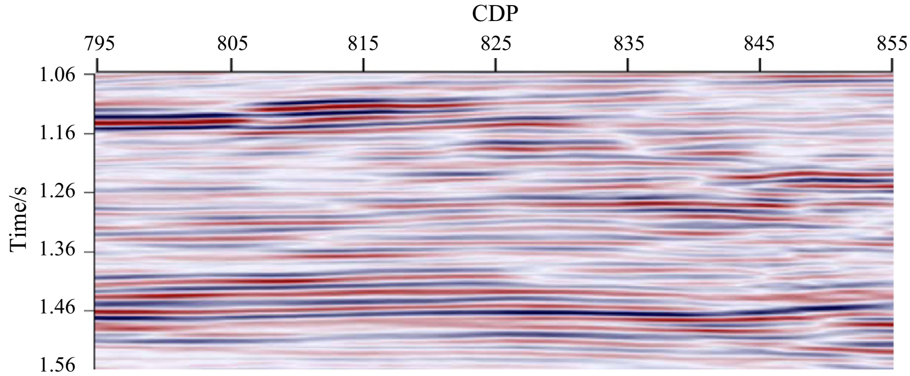

Fig. 6. Seismic section of Inline 645.

where **E**, **R**, **A**, and **B** are diagonal weighting matrices whose diagonal elements are

$$E_{ii} = \frac{1}{|e_i|} \tag{24}$$

$$R_{ii} = \frac{1}{|r_i|} \tag{25}$$

$$A_{ii} = \frac{1}{|a_i|} \tag{26}$$

$$B_{ii} = \frac{1}{|g_i|}. \tag{27}$$

Setting $\nabla f = \mathbf{0}$, we have

$$\left(\mathbf{G}^T\mathbf{E}\mathbf{G} + \alpha\mathbf{R} + \alpha\mathbf{T}_1^T\mathbf{A}\mathbf{T}_1 + \beta\mathbf{C}^T\mathbf{B}\mathbf{C}\right)\mathbf{r} = \mathbf{G}^T\mathbf{E}\mathbf{d} + \beta\mathbf{C}^T\mathbf{B}\boldsymbol{\xi}. \tag{28}$$

Since the elements in matrices **E**, **R**, **A**, and **B** depend on **r**, Eq. (28) is a nonlinear system and cannot be solved directly.

Generally, finding the inverse solution is complicated. One efficient way is iteratively re-weighted least square, or IRLS. IRLS is an iterative algorithm to find the appropriate weights and solve the inverse problem. Starting from an initial parameter vector $\mathbf{r}^0$, the corresponding diagonal weighting matrices **E**, **R**, **A** and **B** can be constructed by Eqs. (24), (25), (26) and (27). Then, solve Eq. (28) by conjugate gradient method (Tarantola, 2005) to obtain a renewed parameter vector $\mathbf{r}^1$ and corresponding weighting matrix. This process is repeated until the following convergence condition is satisfied

$$\frac{\left\|\mathbf{r}^{k+1} - \mathbf{r}^k\right\|_2}{1 + \left\|\mathbf{r}^{k+1}\right\|_2} < \zeta \tag{29}$$

in which ζ is a tolerance value.

If any element in **e**, **r**, **a** and **g** is non-differentiable, the corresponding tolerances ε_1, ε_2, ε_3, ε_4 can be preset, let

$$E_{ii} = \begin{cases} \dfrac{1}{|e_i|} & |e_i| > \varepsilon_1 \\ \dfrac{1}{\varepsilon_1} & |e_i| < \varepsilon_1 \end{cases} \tag{30}$$

$$R_{ii} = \begin{cases} \dfrac{1}{|r_i|} & |r_i| > \varepsilon_2 \\ \dfrac{1}{\varepsilon_2} & |r_i| < \varepsilon_2 \end{cases} \tag{31}$$

$$A_{ii} = \begin{cases} \dfrac{1}{|a_i|} & |a_i| > \varepsilon_3 \\ \dfrac{1}{\varepsilon_3} & |a_i| > \varepsilon_3 \end{cases} \tag{32}$$

$$B_{ii} = \begin{cases} \dfrac{1}{|g_i|} & |g_i| > \varepsilon_4 \\ \dfrac{1}{\varepsilon_4} & |a_i| < \varepsilon_4 \end{cases}. \tag{33}$$

With these modifications, the inversion process will stably converge to a reasonable solution.

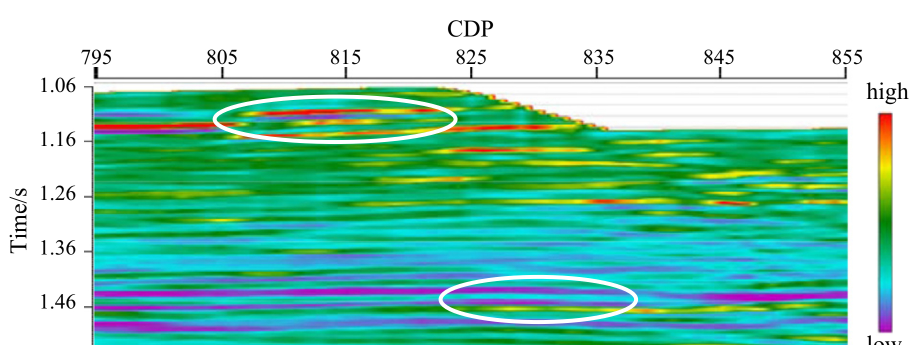

Fig. 7. The inversion section for Inline 645.

From the solution of Eq. (28), we can get the estimated parameters of P-wave impedance via the following formula (Zhang, Yin and Zhang, 2011)

$$m(t) = m(t_0) \exp\left[2\int_{t_0}^{t} r(\eta) d\eta\right]. \tag{34}$$

4. Numerical and real seismic data examples

First, we used a one-dimensional model to analyze the feasibility, stability and anti-noise characteristics of the proposed method. The impedance model and corresponding true reflection coefficients are shown in Figs. 3(a) and 2(a), respectively. The seismic data shown in Fig. 1(a) is generated by convolving the reflection coefficients in Fig. 2(a) with a Ricker wavelet whose dominant frequency is 35 Hz. Then, 10% Gaussian random noise and some outliers are added into the seismic data, as shown in Fig. 1(b). In Fig. 1(b), the red points represent the outliers. Three inversion methods are performed using the tainted seismic data shown in Fig. 1(b). The first inversion method is based on L1-norm misfit function and TVR, the second one is based on L1-norm misfit function but no TVR, and the third one is on L2-norm misfit function. The a priori regularization, which is plotted in green in the latter three panels of Fig. 3, is the same for all the three inversions. The corresponding reflection coefficient inversion results and impedance inversion results are plotted with blue curves in Figs. 2 and 3, respectively. From Figs. 2 and 3 we can see that the L2-norm-based method gives wrong results, while the estimated parameters from the L1-norm-based method are closer to the true model. Furthermore, comparing the inversion results based on L1-norm misfit function, the inversion with TVR obviously does a better job in recovering the discontinuous locations or layer boundaries than the one without TVR.

In addition, we performed inversion using a two-dimensional model. The 2-D impedance model is shown in Fig. 5(a), whose corresponding seismic data is shown in Fig. 4. Fig. 4 is generated by the same way as in the 1-D model experiment. The seismic data in Fig. 4 is used to perform L1-norm-based inversion with TVR. The inversion result is shown in Fig. 5(b). From the comparison between inversion result in Fig. 5(b) and true model in Fig. 5(a), we can see that the inverted impedance is in accordance with the true model data.

Next, we implemented real seismic data inversion to verify the applicability of the proposed method. The real seismic data cube contains 141 in-lines from 525 to 665, and 61 cross-lines from 795 to 855, respectively. Fig. 6 shows the inline 645 seismic section. After environmental correction, scale equalization processing for the well log data in this survey, we build the priori impedance model under the constraints of geologic horizons. Then the L1-norm misfit function based inversion with TVR is performed on the real seismic data. Fig. 7 shows the inverted vertical section for inline 645. It can be seen from Fig. 7 that the inversion result is reasonable and the interfaces between the layers are obvious (note the bed interface within the oval).

In order to compare with the traditional L2-norm based inversion results, we extract the seismic data at two locations, CDP813 and CDP830, from inline 645 shown in Fig. 6, to perform inversion based on our proposed method and L2-norm method, respectively. The inversion results are shown in Fig. 8, in which the blue curves are the inverted impedance parameters by our proposed method, the red curves are those inverted by the method based on L2-norm misfit function without TVR, and the green curves are initial models. We can see from Fig. 8 that: i) the initial model for seismic inversion is a smooth curve; ii) the interfaces between the layers are obviously drawn in blue, indicating abrupt changes between two layers; while the curves drawn in red represent smooth changes; and iii) the vertical resolution drawn in blue is better than that drawn in red. In addition, the inverted impedance parameters by our proposed method are compared with the real log data at locations CDP813 and CDP830. The comparisons are shown in Fig. 9, in which the blue curves are the inverted impedance parameters by the proposed method, the red curves are the real log data. It can be seen from these comparisons in Fig. 9 that the inverted impedance parameters by L1-norm-based inversion with TVR are well matched with the real well log data.

Further evidence to support the proposed inversion method is shown from the time slices. Fig. 10 shows a time slice from the inversion result at time 1.195 s, from which the layer boundaries are easy to be delineated.

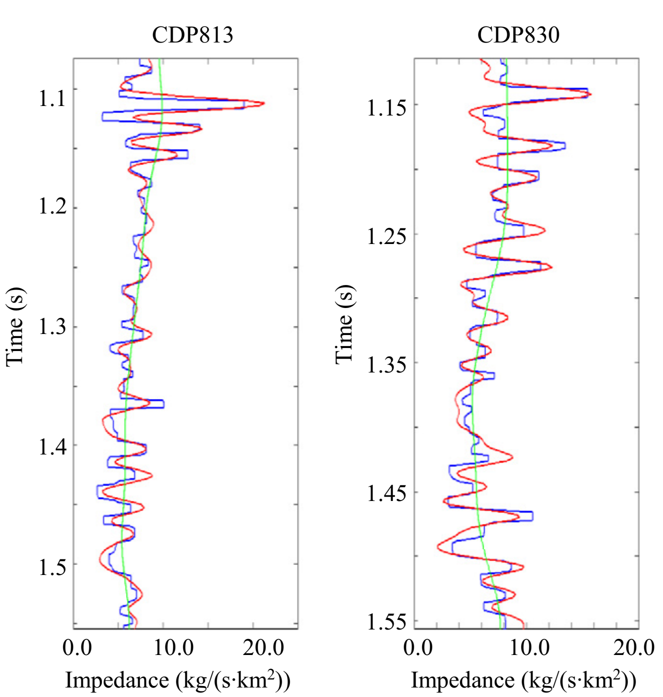

Fig. 8. Comparison between L1 norm inversion results and L2 norm inversion results.

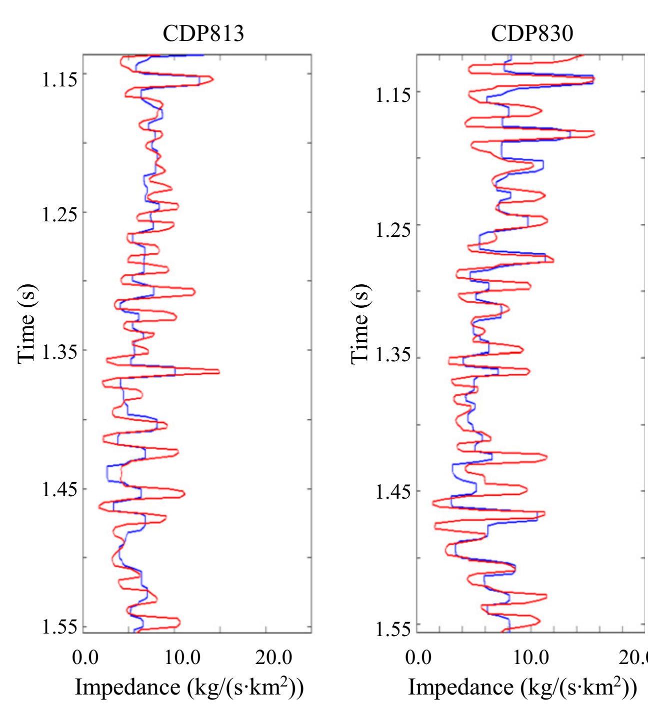

Fig. 9. Comparison between L1 norm inversion results and the real well log data.

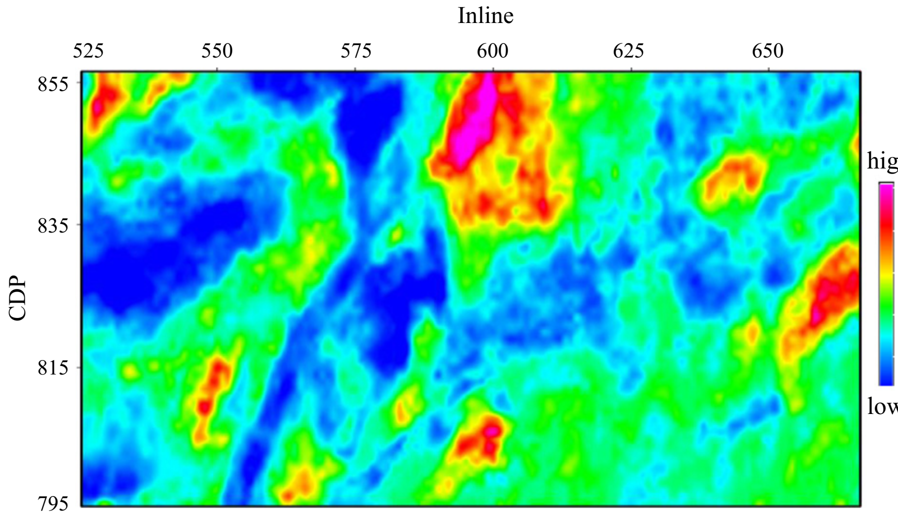

Fig. 10. The time slice of inversion result at 1.195 s.

5. Discussions

Picking the regularization parameters are an important step in inversion algorithm, and the mostly used method to pick the good regularization parameters is the L curve criterion (Hansen, 1992). Look at the Eq. (7) for example. Note as δ decreases, the set of feasible solutions becomes “flatter” and “smoother”, and the minimum of $||\mathbf{G}(\mathbf{m}) - \mathbf{d}||_1$ increases. As δ is adjusted, the curve of optimal values of $||\mathbf{T}_i\mathbf{m}||_1$ and $||\mathbf{G}(\mathbf{m}) - \mathbf{d}||_1$ can be traced out. When this curve is plotted on a log–log coordinate, it often appears as an “L” shape characteristic for linear inverse problems. The sharpness of the corner varies for different inverse problems, but it is always well defined. The best value of regularization parameters closest to the corner of the L-curve is picked as the optimum solution. An alternative way for picking a set of good regularization parameters is generalized cross-validation (GCV) (Craven and Wahba, 1979; Golub and von Matt, 1997; Wahba, 1990). This method provides a theoretical justification to pick the regularization parameters.

Here we discuss another alternative method to pick the regularization parameters. If there are some well logs in a seismic survey, we can do quality control at the well locations. The specific process is adjusting the regularization parameters and picking one set of parameters which can estimate the inversion solution with the best fitness to the real well log data.

For the advantages and disadvantages of the proposed method, as we can see from the above sections, the advantage of the proposed method is its robustness for overcoming the influence of outliers which may arise from the procedural measurement error, or other reasons, in the seismic data. In addition, with the TVR terms in objective function, the proposed inversion method can estimate the discontinuities which are desirable in geologic environments with abrupt changes in impedance, such as carbonate caves and salt bodies. However, there are some disadvantages in the proposed method. The biggest one is that the inverse problem is nonlinear, hence leading to a higher degree of computational complexity compared with the L2-norm-based method.

6. Conclusions

If there are outliers in seismic data, it is necessary to perform seismic inversion in a way that minimizes their effects on the estimated parameters. However, the seismic inversion based on L2-norm misfit function is highly susceptible to even small numbers of inconsistent seismic observations.

In this paper, we proposed a seismic inversion method based on L1-norm misfit function with total variation regularization. This method is appropriate to solve the inverse problem when outliers exist in the seismic data and discontinuities such as layer interfaces need to be clearly delineated. In order to ensure that the inverted parameters contain low frequency information, a priori information constraint can be added as a regularization term into the objective function. The inverse problem is solved by the iteratively re-weighted least square strategy. As we show in this paper, the inversion method is successfully tested on noisy synthetic seismic data with outliers and real seismic data. The test results tell us that the inversion method based on L1-norm misfit function with TVR can even better reveal the abrupt changes between sedimentary layers and more coincident with true model or real log data. Of course, there are some alternative techniques to find the favorable regularization parameters. In a sense, a set of good regularization parameters is the key of the seismic inversion process.

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